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Secondary flow in non-isothermal viscoelastic parallel-plate flow

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Abstract. The non-isothermal, parallel-plate flow of a viscoelastic fluid is investigated. Using the singular perturbation method the problem for the stream function χ is reduced to a non-homogeneous boundary-value problem for the biharmonic operator in an infinite strip. An exact solution is then obtained using Fourier integral transforms. Our analysis shows the existence of thermally induced recirculating eddies, even when inertia is neglected.

Key words: parallel-plate, secondary flow, viscoelastic, viscous heating

1. Introduction

The non-isothermal flow of a viscoelastic fluid between two coaxial disks is considered. In viscometric applications, this flow forms the basis for obtaining rheological properties such as the viscosity and normal stress coefficients of a fluid. These measurements are based on the assumption that the flow is steady, purely azimuthal and isothermal. Any significant departure from this base flow could lead to appreciable errors in the predicted material properties [1, p. 226] and [2, p. 80]. The assumption that the flow is one-dimensional is valid, however, only if inertia is neglected. The presence of inertial forces gives rise to the formation of a recirculating eddy motion which is superimposed on the purely azimuthal flow [3–6]. For flows with very small Reynolds number, and hence very small inertial forces, the secondary flow is very weak and therefore may be neglected. Under isothermal conditions, viscoelastic fluids behave in a qualitatively similar fashion when sheared between coaxial parallel plates provided that the shear rate is not too large [7,8]. At sufficiently large shear rates, even the creeping flow of viscoelastic fluids is known to be subject to purely elastic instabilities [9–12]. Above some critical shear rate there is a Hopf bifurcation to a time-dependent flow and the formation of secondary roll cells [13–15]. Thus, above some critical shear rate, prediction of rheological properties based on the basic torsional-flow assumption is no longer accurate.

When a fluid is sheared continuously between two plates, viscous dissipation can cause significant temperature changes in the fluid. This is especially true for polymeric liquids which are poor conductors and have strongly temperature-dependent viscosities. It is well known that viscous heating can lead to significant errors in viscometric measurements [1,16,17]. The effect of viscous dissipation on torsional flow of Newtonian and power-law fluids has been analyzed in [18–20]. For viscoelastic fluids governed by the Oldroyd-B model, similar analyses have been reported in [21,22]. In all these studies the flow is assumed to be purely azimuthal and with the exception of [18,21], the flow domain is assumed to be infinite. In [18], the velocity profile was assumed to be isothermal and the energy equation which is linear was solved numerically by a variational method. In [21] perturbation methods were used to solve the problem for both the temperature and non-isothermal velocity profiles. In this paper we show that for non-isothermal viscoelastic torsional flow the one-dimensional flow assumption also fails. In particular, we show the existence of secondary vortices even in the absence of inertia. This suggests that corrections due to viscous heating may be necessary, not only for the viscosity but also the normal-stress coefficients. To the best of our knowledge, there is no theoretical analysis on the existence or otherwise of secondary flow in non-isothermal, creeping viscoelastic torsional flow.

2. Governing equations

We consider steady, axisymmetric, non-isothermal flow of an incompressible viscoelastic fluid between two coaxial disks of radius a, separation h, and aspect ratio $\alpha \equiv h/a$. The fluid is subjected to a shearing motion by rotating the top plate at a constant angular speed ω.

In dimensional form, the equation governing the steady flow of an incompressible viscoelastic fluid are the continuity equation [1, Chapter 7]

$$\nabla \cdot \tilde{\mathbf{v}} = 0, \tag{2.1}$$

and the momentum equation, which in the absence of inertia is given by

$$0 = -\nabla \tilde{p} + \nabla \cdot \tilde{\boldsymbol{\sigma}}, \tag{2.2}$$

where $\tilde{\mathbf{v}}$ is the velocity, $\tilde{p}$ is the pressure and $\tilde{\boldsymbol{\sigma}}$ is the stress tensor. The stress tensor can be written as

$$\tilde{\boldsymbol{\sigma}} = \eta_s \dot{\tilde{\boldsymbol{\gamma}}} + \tilde{\boldsymbol{\tau}},$$

where $\dot{\tilde{\boldsymbol{\gamma}}}$ is the rate-of-strain tensor, $\tilde{\boldsymbol{\tau}}$ is the extra stress arising from the polymer and η_{s} is the solvent viscosity. In this paper we will consider a viscoelastic fluid described by the Oldroyd-B constitutive model. A non-isothermal version of this model based on the pseudo-time hypothesis gives the following equation for the extra stress $\tilde{\boldsymbol{\tau}}$ [23, Chapter 11], [24]

$$\tilde{\boldsymbol{\tau}} + \lambda[\tilde{\mathbf{v}} \cdot \boldsymbol{\nabla} \tilde{\boldsymbol{\tau}} - (\boldsymbol{\nabla}\tilde{\mathbf{v}})^{\mathrm{T}} \tilde{\boldsymbol{\tau}} - \tilde{\boldsymbol{\tau}} \boldsymbol{\nabla} \tilde{\mathbf{v}} - \tilde{\boldsymbol{\tau}} \tilde{\mathbf{v}} \cdot \boldsymbol{\nabla} \log \tilde{T}] = \eta_p \dot{\tilde{\boldsymbol{\gamma}}}, \tag{2.3}$$

where λ is the relaxation time and η_p is the polymer viscosity. If $\tilde{T}$ is constant, we recover the isothermal Oldroyd-B equation. Neglecting inertia, we obtain the energy equation for the temperature $\tilde{T}$as follows:

$$0 = \kappa \nabla^2 \tilde{T} + \tilde{\boldsymbol{\sigma}} : \nabla \tilde{\mathbf{v}}, \tag{2.4}$$

where κ is the conductivity.

The boundary conditions are no slip at the plates and constant temperature $\tilde{T}_w$ at the plates which, for simplicity, we assume is equal to the ambient temperature. At the centerline $\tilde{r} = 0$ we require symmetry while, at the free surface $\tilde{F}(\tilde{r}, \tilde{z}) \equiv \tilde{f}(\tilde{z}) - \tilde{r} = 0$, we have the following interface conditions

$$\mathbf{v} \cdot \nabla \tilde{F} = 0, \tag{2.5}$$

and

$$(\tilde{p} - p_{\mathrm{a}})\mathbf{n} + \tilde{\boldsymbol{\sigma}} \cdot \mathbf{n} = \gamma \mathcal{H} \mathbf{n}. \tag{2.6}$$

Finally we require that the free surface remains pinned to the plates

$$\tilde{f}(0) = \tilde{f}(h) = a. \tag{2.7}$$

Here p_a is the atmospheric pressure, γ is the surface tension, $\mathbf{n}$ is the outward normal and $\mathcal{H}$ is the mean curvature of the surface. The relaxation time λ, and the viscosities η_s and η_p are strongly temperature-dependent and will be modelled by a Nahme type law [17,20,26,27] as follows:

$$\lambda = \lambda_0 \frac{\tilde{T}_0}{\tilde{T}} e^{-\Theta}, \quad \eta_s = \eta_{s0} e^{-\Theta}, \quad \eta_p = \eta_{p0} e^{-\Theta}.$$

where $\Theta = \delta(\tilde{T} - \tilde{T}_0)/\tilde{T}_0$ is a dimensionless temperature. Here λ_0 denotes the relaxation time at a reference temperature $\tilde{T}_0$, while η_{s0} and η_{p0} are, respectively, the solvent and polymer viscosities at the reference temperature. The quantity δ is a dimensionless thermal-sensitivity parameter of the fluid. Typically δ is very large so that even small deviations of the temperature from the reference temperature lead to O(1) changes in Θ. For polyisobutylene-based liquids, experiments suggest a sensitivity parameter $\delta = 20$ [25], and for polystyrene-based elastic liquids $\delta = 60$ [11]. The temperature dependence of the material functions can also be modelled by an Arrhenius-type law of the from $\exp[-\delta(\tilde{T} - \tilde{T}_0)/\tilde{T}]$. For the range of temperatures relevant to the flows of interest the two models do not differ very much [26–28]; therefore, in order to obtain analytical solutions, we will use the more mathematically tractable Nahme law.

We will adopt a cylindrical coordinate system $(\tilde{r}, \theta, \tilde{z})$. The equations are non-dimensionalized as follows:

$$\tilde{r} = ar, \quad \tilde{z} = hz, \quad \tilde{\mathbf{v}} = a\omega(u, w, \alpha v),$$
$$(\tilde{p} - p_a, \tilde{\boldsymbol{\tau}}) = \eta_0\omega(p, \boldsymbol{\tau}), \quad \tilde{T} = \tilde{T}_0 T, \quad \tilde{f} = af.$$

Here $\eta_0 \equiv \eta_{s0} + \eta_{p0}$ is the zero-shear-rate viscosity of the fluid at the reference temperature. The important parameters for this problem are the Deborah number $\mathrm{De} = \lambda_0\omega e^{-\vartheta_w}$, the retardation parameter $\beta = \eta_{p0}/\eta_0$, the Nahme-Griffith number $\mathrm{Na} = (\delta\eta_0 a^2\omega^2 e^{-\vartheta_w})/(\kappa\tilde{T}_0)$ and the capillary number $C_a = (e^{-\theta_w}\eta_0 a\omega)/\gamma$ where ϑ_w is the reduced wall temperature.

3. One-dimensional flow: a necessary condition

We claim that for viscoelastic fluids there is no solution with $u = v = 0$ in flows in which viscous heating is significant. Assume for the moment that $u = v = 0$, then the governing equations reduce to

$$r\frac{\partial p}{\partial r} = -\tau_{\theta\theta}, \tag{3.1}$$

$$\frac{\partial w}{\partial z^2} + \alpha^2\left(\frac{\partial w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{w}{r^2}\right) = \frac{\partial\Theta}{\partial z}\frac{\partial w}{\partial z} + \alpha^2\frac{\partial\Theta}{\partial r}\left(\frac{\partial w}{\partial r} - \frac{w}{r}\right), \tag{3.2}$$

$$\frac{\partial p}{\partial z} = 0, \tag{3.3}$$

and

$$\frac{\partial\Theta}{\partial z^2} + \alpha^2\left(\frac{\partial\Theta}{\partial r^2} + \frac{1}{r}\frac{\partial\Theta}{\partial r}\right) = -\mathrm{Na}\,e^{-\Theta}\left[\left(\frac{\partial w}{\partial z}\right)^2 + \alpha^2\left(\frac{\partial w}{\partial r} - \frac{w}{r}\right)^2\right]. \tag{3.4}$$

The non-zero stresses are given, respectively, by

$$\tau_{\theta,z} = \frac{\beta}{\alpha}e^{-\Theta}\frac{\partial w}{\partial z}, \quad \tau_{r\theta} = \beta e^{-\Theta}\left(\frac{\partial w}{\partial r} - \frac{w}{r}\right), \tag{3.5–6}$$

and

$$\tau_{\theta\theta} = 2\mathrm{De}\beta \mathrm{e}^{-2\Theta}\left[\frac{1}{\alpha^2}\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial r} - \frac{w}{r}\right)^2\right]. \tag{3.7}$$

The boundary conditions are

$$w = 0, \quad \Theta = \vartheta_w \quad \text{on } z = 0, \tag{3.8}$$

$$w = r, \quad \Theta = \vartheta_w \quad \text{on } z = 1, \tag{3.9}$$

$$\frac{\partial w}{\partial r} - \frac{w}{r} = 0, \quad \Theta = \vartheta_w \quad \text{on } r = 1 \tag{3.10}$$

and the symmetry condition

$$w = 0, \quad \text{when } r = 0. \tag{3.11}$$

From Equations (3.1) and (3.3) a necessary condition for a one-dimensional solution to exist is the compatibility condition

$$\frac{\partial \tau_{\theta\theta}}{\partial z} = 0. \tag{3.12}$$

In other words the normal stress $\tau_{\theta\theta}$ must be a function of r only. For a Newtonian fluid the normal stress is zero; therefore such a one-dimensional flow exists. The situation is, however, different if the fluid is viscoelastic. The compatibility condition (3.12) imposes an additional constraint on the velocity so that the system is over-determined. For $\beta\mathrm{De} \neq 0$, Equation (3.12) can be reduced to the following

$$\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r^2} + \left(\frac{\partial w}{\partial r} - \frac{w}{r}\right)\left(\frac{2}{r}\frac{\partial w}{\partial z} - \frac{\partial^2 w}{\partial z \partial r}\right) = \left(\frac{\partial w}{\partial r} - \frac{w}{r}\right)\frac{\partial w}{\partial z}\frac{\partial \Theta}{\partial r} - \left(\frac{\partial w}{\partial r} - \frac{w}{r}\right)^2 \frac{\partial \Theta}{\partial z}. \tag{3.13}$$

For isothermal flow the solution is $w = rz$ and $\Theta = \vartheta_w$ and Equation (3.13) is satisfied. If the flow is non-isothermal, the over-determined system (3.2), (3.4) and (3.13) cannot be satisfied in general and so for viscoelastic fluids viscous heating can be expected to lead to secondary flow, the mechanism responsible being the normal stress gradient. Such normal (hoop) stress stratification has also been identified as the mechanism driving instability in viscoelastic Taylor–Couette flow at zero Reynolds number [29]. For non-isothermal flow, Equations (3.2) and (3.4) must be solved simultaneously for the velocity and temperature. For an exact solution in an unbounded domain see [19, 26, 30]. In [21], this problem was solved by a regular perturbation expansion in Na, while the coupling with the stream function was disregarded. These solutions do not satisfy the compatibility equation (3.13). In this paper, we discuss conditions under which the uncoupling is valid. We also complete the analysis by solving the equation governing the stream function.

4. Secondary flow

We introduce a stream function χ defined by

$$u = \frac{1}{r}\frac{\partial \chi}{\partial z}, \quad v = -\frac{1}{r}\frac{\partial \chi}{\partial r}.$$

In order to obtain exact analytical solutions, we will seek a solution in powers of the Nahme–Griffith number Na as follows:

$$\chi = \mathrm{Na}\,\chi' + \cdots, \quad w = rz + \mathrm{Na}\,w' + \cdots, \quad \Theta = \vartheta_w + \mathrm{Na}\,\Theta' + \cdots,$$
$$\tau_{rr} = \mathrm{Na}\,\tau'_{rr} + \cdots, \quad \tau_{r\theta} = \mathrm{Na}\,\tau'_{r\theta} + \cdots, \quad \tau_{rz} = \mathrm{Na}\,\tau'_{rz} + \cdots,$$
$$\tau_{\theta\theta} = \bar{\tau}_{\theta\theta} + \mathrm{Na}\,\tau'_{\theta\theta} + \cdots, \quad \tau_{\theta z} = \bar{\tau}_{\theta z} + \mathrm{Na}\,\tau'_{\theta z} + \cdots, \quad \tau_{zz} = \mathrm{Na}\,\tau'_{zz} + \cdots,$$
$$p = \bar{p} + \mathrm{Na}\,p', \quad f = 1 + \mathrm{Na}\,\alpha^2 f'.$$

The barred quantities are the isothermal pressure and stresses given by

$$\bar{\tau}_{\theta z} = \frac{\beta \mathrm{e}^{-\vartheta_w}}{\alpha} r, \quad \bar{\tau}_{\theta\theta} = 2\frac{\beta \mathrm{e}^{-\vartheta_w}\mathrm{De}}{\alpha^2} r^2, \quad \bar{p} = (1 - r^2)\frac{\mathrm{e}^{-\vartheta_w}\beta \mathrm{De}}{\alpha^2} + C_a.$$

The perturbation approach is similar to that used by Joseph [5]. Since this expansion is a linear approximation in Na, it will be most accurate when Na is small. The experiments of Rothstein and McKinley [11] spanned a range of Na from 10^{-3} to 1·0. Turian and Bird [19] showed that, even for Na $\simeq$ 0·1, calculated values of the torque in a cone-plate viscometer reveal an appreciable difference from experimentally measured values. In slit-die viscometry, Ko and Lodge [16] have found that the relevant values of Na are typically less than 0·1. So although the expansion in Nahme-Griffith number is used to facilitate analytical calculations, there are practical applications where Na is small. Let $\zeta' = w' - \beta \mathrm{De}\chi'_r$; then to leading order in Na, the equations for χ', ζ' and Θ' (after dropping the primes) reduce to

$$\frac{\partial^4\chi}{\partial z^4} + 2\alpha^2\left(\frac{\partial^4\chi}{\partial z^2\partial r^2} - \frac{1}{r}\frac{\partial^3\chi}{\partial z^2\partial r}\right) + \alpha^4\left(\frac{\partial^4\chi}{\partial r^4} - \frac{2}{r}\frac{\partial^3\chi}{\partial r^3} + \frac{3}{r^2}\frac{\partial^2\chi}{\partial r^2} - \frac{3}{r^3}\frac{\partial\chi}{\partial r}\right)$$
$$= 4r\beta\mathrm{De}\left(\frac{\partial^2\zeta}{\partial z^2} - r\frac{\partial\Theta}{\partial z}\right) + \beta\mathrm{De}^2(4\beta - 6)r\frac{\partial^3\chi}{\partial z^2\partial r}, \tag{4.1}$$

$$\frac{\partial^2\zeta}{\partial z^2} + \alpha^2\left(\frac{\partial^2\zeta}{\partial r^2} + \frac{1}{r}\frac{\partial\zeta}{\partial r} - \frac{\zeta}{r^2}\right) = r\frac{\partial\Theta}{\partial z} - 4\beta\mathrm{De}\frac{1}{r}\frac{\partial^2\chi}{\partial z^2}, \tag{4.2}$$

and

$$\frac{\partial^2\Theta}{\partial z^2} + \alpha^2\left(\frac{\partial^2\Theta}{\partial r^2} + \frac{1}{r}\frac{\partial\Theta}{\partial r}\right) = -r^2. \tag{4.3}$$

The boundary conditions become

$$\chi = 0, \quad \frac{\partial\chi}{\partial z} = 0, \quad \zeta = 0, \quad \Theta = 0 \quad \text{on } z = 0,\ 1, \tag{4.4}$$

On $r = 1$, we have

$$\chi = 0, \quad \frac{\partial^2\chi}{\partial r^2} - \frac{\partial\chi}{\partial r} = 0, \quad \Theta = 0 \tag{4.5}$$

$$\frac{\partial\zeta}{\partial r} - \frac{\zeta}{r} - \frac{\partial f}{\partial z} + \alpha^2(1 - \beta)zf = 0, \tag{4.6}$$

and at the centerline $r = 0$, we have

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial r}\right) = o(1), \quad \zeta = 0, \quad |\Theta| < \infty. \tag{4.7}$$

Lastly, the free-surface deflection satisfies the equation

$$\frac{\mathrm{d}^3 f}{\mathrm{d}z^3}+\alpha^2\frac{\mathrm{d}f}{\mathrm{d}z}-\tilde{C}_a\left[\alpha^3\frac{\partial^3\chi}{\partial r^3}+3\alpha\frac{\partial^3\chi}{\partial r\partial z^2}\right]=0. \tag{4.8}$$

Here $\tilde{C}_a=C_a\mathrm{e}^{-\theta_w}/\alpha=(\eta_0\omega a^2\mathrm{e}^{-\theta_w})/(h\gamma)$. The analysis presented in this paper can be applied to other choices of the boundary conditions as well. In particular, we will discuss later the effect of applying an insulated boundary condition at the free surface, *i.e.*, $\partial\Theta/\partial r=0$ on $r=1$.

4.1. Newtonian fluids

For a Newtonian fluid $\beta\mathrm{De}=0$ so that the right-hand side of Equation (4.1) is zero. The stream function χ satisfies a homogeneous equation with homogeneous boundary conditions which admits of only the trivial solution $\chi\equiv 0$. Hence, for a Newtonian fluid, no secondary flow is generated.

4.2. Non-Newtonian fluids

For viscoelastic fluids $\beta\mathrm{De}\neq 0$ so that Equation (4.1) is non-homogeneous. As remarked above, $\chi\to 0$ as $\beta\mathrm{De}\to 0$; therefore we re-scale as follows:

$$\chi=\beta\mathrm{De}\hat{\chi},\quad f=\beta\mathrm{De}\hat{f}.$$

Substituting in Equations (4.1–4.3) and dropping the hats, we obtain

$$\frac{\partial^4\chi}{\partial z^4}+2\alpha^2\left(\frac{\partial^4\chi}{\partial z^2\partial r^2}-\frac{1}{r}\frac{\partial^3\chi}{\partial z^2\partial r}\right)+\alpha^4\left(\frac{\partial^4\chi}{\partial r^4}-\frac{2}{r}\frac{\partial^3\chi}{\partial r^3}+\frac{3}{r^2}\frac{\partial^2\chi}{\partial r^2}-\frac{3}{r^3}\frac{\partial\chi}{\partial r}\right)$$
$$=4r\left(\frac{\partial^2\zeta}{\partial z^2}-r\frac{\partial\Theta}{\partial z}\right)+\beta\mathrm{De}^2(4\beta-6)r\frac{\partial^3\chi}{\partial z^2\partial r}, \tag{4.9}$$

$$\frac{\partial^2\zeta}{\partial z^2}+\alpha^2\left(\frac{\partial^2\zeta}{\partial r^2}+\frac{1}{r}\frac{\partial\zeta}{\partial r}-\frac{\zeta}{r^2}\right)=r\frac{\partial\Theta}{\partial z}-4\beta^2\mathrm{De}^2\frac{1}{r}\frac{\partial^2\chi}{\partial z^2}, \tag{4.10}$$

and

$$\frac{\partial^2\Theta}{\partial z^2}+\alpha^2\left(\frac{\partial^2\Theta}{\partial r^2}+\frac{1}{r}\frac{\partial\Theta}{\partial r}\right)=-r^2. \tag{4.11}$$

The boundary conditions remain the same, except for (4.6), which becomes

$$\frac{\partial\zeta}{\partial r}-\frac{\zeta}{r}-\beta\mathrm{De}\frac{\partial f}{\partial z}+\beta\mathrm{De}\alpha^2(1-\beta)zf=0, \tag{4.12}$$

5. Perturbation expansions

In rheometric applications the aspect ratio α is small, typically less than 0·1 [1, Chapter 10]. However, the limit $\alpha\to 0$ of Equations (4.9–4.11) is singular since boundary conditions at the free surface cannot be satisfied. In this section we will solve the equations using the method of matched asymptotic expansions. For the inner expansion we will introduce a stretched variable $\xi=(1-r)/\alpha$ so that $\partial/\partial r=-\alpha^{-1}\partial/\partial\xi$. Consequently, the term χ_{rzz} on the right-hand side of Equation (4.9) becomes $-\alpha^{-1}\chi_{\xi zz}$. Therefore, in order to have the correct balance of the terms on both sides of the equation, we must have $\beta \mathrm{De}^2/\alpha$ at most O(1). Linear stability analysis has shown that viscoelastic torsional flow tends to be destabilized by elastic forces [9, 13]. Specifically, torsional flow becomes unstable if the Deborah number De is greater than some critical Deborah number $\mathrm{De_c}$, which depends on the aspect ratio α. For $\alpha << 1$, the critical Deborah number scales like $\sqrt{\alpha}$ [10]. Hence, if the Deborah number is greater than $\mathrm{De_c}$, the steady flow becomes unstable and a Hopf bifurcation to a time-dependent solution occurs [15]. Thus, stable steady flow is to be expected only if De is sufficiently small. In the following analysis we define a new parameter μ as

$$\mu = \frac{\mathrm{De}^2}{\alpha},$$

where μ is at most O(1) so that $\mathrm{De}^2 = \mathrm{O}(\mu\alpha)$.

5.1. Outer solution

Assume an outer expansion

$$\chi = \chi_0 + \mathrm{O}(\alpha), \quad \zeta = \zeta_0 + \mathrm{O}(\alpha), \quad \Theta = \Theta_0 + \mathrm{O}(\alpha), \quad \mu = \mu_0 + \mathrm{O}(\alpha). \tag{5.1}$$

Taking the limit $\alpha \to 0$ in (4.9–4.11), we obtain

$$\frac{\partial^4 \chi_0}{\partial z^4} = 4r\left(\frac{\partial^2 \zeta_0}{\partial z^2} - r\frac{\partial \Theta_0}{\partial z}\right), \quad \frac{\partial^2 \zeta_0}{\partial z^2} = r\frac{\partial \Theta_0}{\partial z}, \quad \frac{\partial^2 \Theta_0}{\partial z^2} = -r^2. \tag{5.2–4}$$

The boundary conditions to be satisfied are

$$\chi_0 = 0, \quad \frac{\partial \chi_0}{\partial z} = 0, \quad \zeta_0 = 0, \quad \Theta_0 = 0 \quad \text{on} \quad z = 0,\ 1. \tag{5.5}$$

Equations (5.2–4–5.5) have the solution

$$\chi_0 = 0, \quad \zeta_0 = \frac{r^3}{12} z(1-z)(2z-1), \quad \Theta_0 = \frac{r^2}{2} z(1-z). \tag{5.6–8}$$

5.2. Inner solution

Let $\xi = (1-r)/\alpha$ and expand as follows

$$\chi = \chi^i + \mathrm{O}(\alpha), \quad \zeta = \zeta^i + \mathrm{O}(\alpha), \quad \Theta = \Theta^i + \mathrm{O}(\alpha), \quad \mu = \mu_0 + \mathrm{O}(\alpha). \tag{5.9}$$

The inner equations are then given by

$$\nabla^4 \chi^i - \Lambda_0 \frac{\partial^3 \chi}{\partial \xi \partial z^2} = -4\frac{\partial^2 \zeta^i}{\partial \xi^2}, \quad \nabla^2 \zeta^i = \frac{\partial \Theta^i}{\partial z}, \quad \nabla^2 \Theta^i + 1 = 0, \tag{5.10–12}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \xi^2},$$

and

$$\Lambda_0 = \beta \mu_0 (6 - 4\beta).$$

Since $0 \le \beta \le 1$, it follows that $\Lambda_0 \ge 0$. Note that we have used Equation (5.10–12) to simplify the right-hand side of Equation (5.10–12). The boundary conditions are

$$\chi^i = \frac{\partial \chi^i}{\partial z} = 0, \quad \zeta^i = \Theta^i = 0 \quad \text{on} \quad z = 0,\ 1, \tag{5.13}$$

$$\chi^i = \frac{\partial^2 \chi^i}{\partial \xi^2} = 0, \quad \Theta^i = \frac{\partial \zeta^i}{\partial \xi} = 0 \quad \text{on} \quad \xi = 0. \tag{5.14}$$

Lastly, we require that the inner solution be bounded as $\xi \to \infty$ and that matching with the outer solution can be achieved in a suitable overlap region. The meniscus deflection satisfies the equation

$$\frac{\mathrm{d}^3 f}{\mathrm{d}z^3} + \tilde{C}_a \left(\frac{\partial^3 \chi}{\partial \xi^3} + 3 \frac{\partial^3 \chi}{\partial \xi \partial z^2} \right) = 0 \tag{5.15}$$

subject to the boundary conditions

$$f(0) = f(1) = 0. \tag{5.16}$$

Equation (5.15) is third order which is why an additional boundary condition is needed. Since the fluid is incompressible, the free-surface location must satisfy the constraint that the volume of liquid remains constant. Once the stream function χ has been determined, Equation (5.15) can then be solved to obtain the meniscus deflection.

We first solve (5.10–12) and (5.10–12) for ζ^i and Θ^i and substitute in (5.10–12) to obtain a non-homogeneous equation for the stream function χ. From [21], we have for the inner expansion

$$\zeta^i = \frac{8}{\pi^5} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left[A_{mn} \mathrm{e}^{-2n\pi\xi} + B_{mn} \mathrm{e}^{-(2m+1)\pi\xi} + C_{mn} \right] \sin(2n\pi z), \tag{5.17}$$

and

$$\Theta^i = \frac{4}{\pi^3} \sum_{m=0}^{\infty} \left(1 - \mathrm{e}^{-(2m+1)\pi\xi} \right) \frac{\sin[(2m+1)\pi z]}{(2m+1)^3}, \tag{5.18}$$

where

$$A_{mn} = \frac{-2}{(2m+1)[4n^2 - (2m+1)^2]^2}, \quad B_{mn} = \frac{4n}{(2m+1)^2[4n^2 - (2m+1)^2]^2}, \tag{5.19–20}$$

and

$$C_{mn} = \frac{1}{n(2m+1)^2[(2m+1)^2 - 4n^2]}. \tag{5.21}$$

A composite expansion valid everywhere was given by

$$\zeta = rz + \mathrm{Na} \left\{ \frac{r^3}{12} z(1-z)(2z-1) + \frac{8}{\pi^5} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} [A_{mn} \mathrm{e}^{-2n\pi(1-r)/\alpha} + \right.$$
$$\left. + B_{mn} \mathrm{e}^{-(2m+1)\pi(1-r)/\alpha}] \sin(2n\pi z) \right\} + \mathrm{O}(\mathrm{Na}^2), \tag{5.22}$$

and

$$\Theta = \vartheta_w + \mathrm{Na} \left\{ \frac{r^2}{2} z(1-z) - \frac{4}{\pi^3} \sum_{m=0}^{\infty} \frac{\mathrm{e}^{-(2m+1)\pi(1-r)/\alpha}}{(2m+1)^3} \sin[(2m+1)\pi z] \right\} + \mathrm{O}(\mathrm{Na}^2). \tag{5.23}$$

Introduce the change of variables $z = y + 1/2$; then the stream function (with the subscript i dropped) satisfies the following boundary-value problem on the semi-infinite strip $0 \le \xi < \infty$, $|y| \le 1/2$,

$$\frac{\partial^4 \chi}{\partial y^4} + 2\frac{\partial^4 \chi}{\partial y^2 \partial \xi^2} + \frac{\partial^4 \chi}{\partial \xi^4} - \Lambda_0 \frac{\partial^3 \chi}{\partial \xi \partial y^2} = -4\frac{\partial^2 \zeta^i}{\partial \xi^2}, \tag{5.24}$$

together with the boundary conditions

$$\chi = \frac{\partial \chi}{\partial y} = 0 \quad \text{on} \quad y = \pm\frac{1}{2}, \tag{5.25}$$

$$\chi = \frac{\partial^2 \chi}{\partial \xi^2} = 0 \quad \text{on} \quad \xi = 0, \quad \chi \equiv 0 \quad \text{as} \quad \xi \to \infty \tag{5.26}$$

Provided that Λ_0 is not an eigenvalue of the corresponding homogeneous problem, the boundary-value problem (5.24–5.26) has a unique non-trivial solution indicating that secondary flow is to be expected. In [31] it was shown that all the eigenvalues are purely imaginary. Since Λ_0 is real, it follows that a unique solution exists for all Λ_0. In order to prove that secondary flow exists for insulating boundary conditions on the free surface, it will suffice to show that the right-hand side of Equation (5.24) is not trivial. The solution for an insulating boundary condition on the free surface as given in Appendix A shows the existence of a boundary later near the free surface. It should be noted that the boundary-layer terms are of order $O(\alpha)$; therefore the secondary flow is expected to be weaker than that for isothermal boundaries.

Exact analytical solutions for the boundary-value problem (5.24–5.26) cannot easily be found for all values of Λ_0. In general, the problem has to be solved numerically (Olagunju, submitted for publication). However, in the special case $\Lambda_0 = 0$, the problem is amenable to analytical methods. In this paper we will consider only this special case. Note from Equation (5.9) that $\Lambda_0 = 0$ does not imply that $\beta \mathrm{De}^2 = 0$, only that it is $O(\alpha^2)$ and that the stream function χ is then at most $O(\alpha)$. If $\Lambda_0 \neq 0$, χ is at most $O(\sqrt{\alpha})$. For $\Lambda = 0$, the boundary conditions (5.26) suggest that we use Fourier-sine transforms. Define the Fourier-sine transform pair

$$U(k, y) = \int_0^\infty \chi(\xi, y) \sin(k\xi) \mathrm{d}\xi \tag{5.27}$$

and the inverse

$$\chi(\xi, y) = \frac{2}{\pi} \int_0^\infty U(k, y) \sin(k\xi) \mathrm{d}k. \tag{5.28}$$

Taking the Fourier-sine transform of Equation (5.24) and the boundary conditions (5.25), we obtain

$$\frac{\mathrm{d}^4 U}{\mathrm{d}y^4} - 2k^2 \frac{\mathrm{d}^2 U}{\mathrm{d}y^2} + k^4 U = F(k, y) \tag{5.29}$$

and the boundary conditions

$$U = U' = 0 \quad \text{on} \quad y = \pm\tfrac{1}{2}, \tag{5.30}$$

where

$$F(k, y) = -4 \int_0^\infty \frac{\partial^2 \zeta^i}{\partial \xi^2} \sin(k\xi) \mathrm{d}\xi.$$

The solution is

$$U=\sum_{n=1}^{\infty} c_n(k)\left[\sin(2n\pi y)+2n\pi(-1)^n\frac{\cos\left(\frac{k}{2}\right)}{\sin k-k}\sin(ky)-4n\pi(-1)^n\frac{\sin\left(\frac{k}{2}\right)}{\sin k-k}y\cos(ky)\right], \tag{5.31}$$

where

$$c_n(k)=(-1)^{n+1}\frac{32}{\pi^5}\sum_{m=0}^{\infty}\frac{(2n\pi)^2A_{mn}k}{[k^2+(2n\pi)^2]^3}+\frac{(2m+1)^2\pi^2B_{mn}k}{[k^2+(2m+1)^2\pi^2][k^2+(2n\pi)^2]^2}. \tag{5.32}$$

Note that the dependence of U on $F(k,y)$ is represented by the constants A_{mn} and B_{mn} in Equation (5.32). Then the stream function χ is given by the inversion formula

$$\chi(\xi,y)=\frac{2}{\pi}\int_0^{\infty}U(k,y)\sin(k\xi)\mathrm{d}k. \tag{5.33}$$

Substituting for U in (5.33), we obtain

$$\chi=\sum_{n=1}^{\infty}Q_{1,n}(\xi)\sin(2n\pi y)+2n\pi(-1)^nQ_{2,n}(\xi,y)-4n\pi(-1)^n yQ_{3,n}(\xi,y), \tag{5.34}$$

where

$$Q_{1,n}(\xi,y)=\tfrac{2}{\pi}\int_0^{\infty}c_n(k)\sin(k\xi)\mathrm{d}k, \tag{5.35}$$

$$Q_{2,n}(\xi,y)=\tfrac{2}{\pi}\int_0^{\infty}\frac{c_n(k)\cos\left(\frac{k}{2}\right)\sin(ky)\sin(k\xi)}{\sin k-k}\mathrm{d}k, \tag{5.36}$$

$$Q_{3,n}(\xi,y)=\tfrac{2}{\pi}\int_0^{\infty}\frac{c_n(k)\sin\left(\frac{k}{2}\right)\cos(ky)\sin(k\xi)}{\sin k-k}\mathrm{d}k. \tag{5.37}$$

The values of the integrals in (5.35–5.37) were evaluated using residues and are given in Appendix B. Since the outer solution for χ is trivial, it follows that Equation (5.34) is uniformly valid for all values of r. Contour plots of the stream function χ for selected values of α are shown in Figures 1–3. The plots show the existence of recirculating vortices. As α decreases, the secondary flow is localized in a boundary layer near the free surface as expected.

6. Discussion

In this paper, we have analyzed the effect of viscous heating on the torsional flow of a viscoelastic fluid between two parallel plates. As is typical in rheometric applications, we have assumed that the separation h between the plates is much smaller than the radius a of the plates. The fluid is modelled by the Oldroyd-B constitutive equation in which the relaxation time and viscosities are exponential functions of the temperature.

For creeping flow in which the Reynolds number is zero, the flow generated by rotating the upper plate at constant angular speed is one-dimensional. This forms the basis upon which the fluid's viscosity and relaxation time are determined experimentally. The one-dimensional-flow assumption is true for Newtonian and viscoelastic fluids if the flow is isothermal. However, the situation is different if viscous heating is present. While the one-dimensional-flow

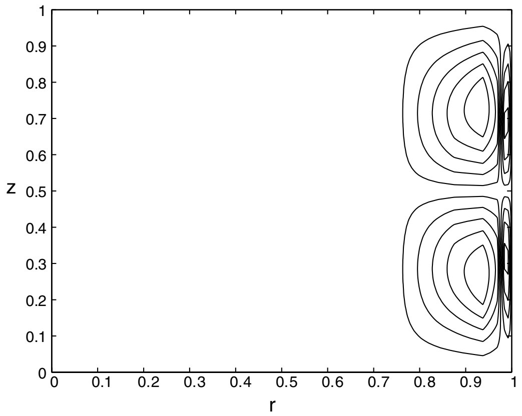

Figure 1. Contour plot of the stream function χ for $\alpha = 0{\cdot}1$.

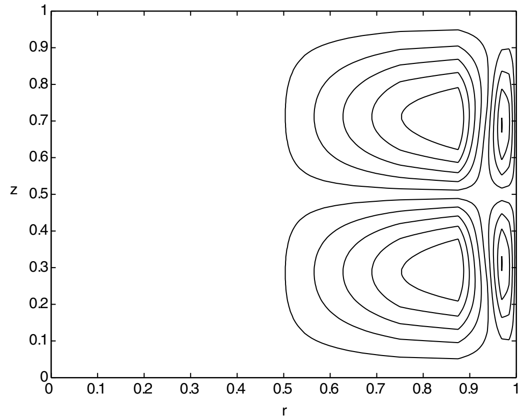

Figure 2. Contour plot of the stream function χ for $\alpha = 0{\cdot}25$.

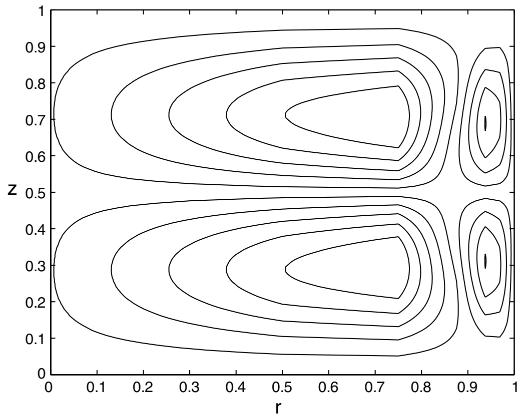

Figure 3. Contour plot of the stream function χ for $\alpha = 0{\cdot}5$.

assumption still holds for Newtonian fluids, we have shown that it is not true if the fluid is viscoelastic. We show that normal (hoop) stress stratification leads to secondary flows with recirculating roll cells normal to the direction of rotation. Thus, when dealing with viscoelastic fluids, proper accounting must be made of the effect of non-isothermally-induced secondary motion in torsional viscometers.

In our analysis we have used a singular perturbation expansion in $\alpha = h/a$ to determine the azimuthal velocity and the stream function χ. We assumed that the Deborah number $\mathrm{De} = \mathrm{O}(\sqrt{\alpha})$ and then show that the stream function $\chi = \mathrm{O}(\beta \mathrm{De})$ where β is the retardation parameter. To leading order in α the equation for the stream function uncouples from the energy equation and the equation for the azimuthal velocity. The stream function is then governed by a non-homogeneous fourth-order equation in which the forcing term arises from the normal-stress gradient. Under normal operating conditions in rheometric applications $\alpha << 1$, so the strength of the secondary flow is at most $\mathrm{O}(\sqrt{\alpha})$.

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Appendix A

In this section we present the solutions of Equations (4.10) and (4.11) when the boundary condition $\Theta=0$ is replaced by the insulated boundary condition $\partial\Theta/\partial r=0$ in Equation (4.5) and the O(De^2) term has been dropped as discussed. The exact solution of Equation (4.11) is

$$\Theta=\vartheta_w+\mathrm{Na}\sum_{m=0}^{\infty}\frac{4}{(2m+1)\pi}\left\{\frac{r^2}{(2m+1)^2\pi^2}+\frac{4\alpha^2}{(2m+1)^4\pi^4}-\right.$$
$$\left.-\frac{2\alpha}{(2m+1)^3\pi^3}\frac{I_0[(2m+1)\pi r/\alpha]}{I_1[(2m+1)\pi/\alpha]}\right\}\sin[(2m+1)\pi z]+\mathrm{O}(\mathrm{Na}^2). \tag{A.1}$$

Here $I_n(z)$ is the modified Bessel functions of order n. For $\alpha<<1$ we obtain

$$\Theta=\vartheta_w+\mathrm{Na}\left\{\frac{r^2}{2}z(1-z)-8\alpha\sum_{m=0}^{\infty}\frac{\mathrm{e}^{-(2m+1)\pi(1-r)/\alpha}}{(2m+1)^4\pi^4}\sin[(2m+1)\pi z]\right\}+\mathrm{O}(\mathrm{Na}^2). \tag{A.2}$$

The solution given in Equation (A.2) can be obtained from Equation (A.1) by using the asymptotic properties of the modified Bessel functions or directly from Equation (4.11) using the singular perturbation method. Similarly, we obtain for ζ the solution

$$\zeta=rz+\mathrm{Na}\left\{\frac{r^3}{12}z(1-z)(2z-1)+\alpha\sum_{n=0}^{\infty}\left[E_{mn}\mathrm{e}^{-2n\pi(1-r)/\alpha}+F_{mn}\right]\sin(2n\pi z)\right\}+$$
$$+\mathrm{O}(\mathrm{Na}^2), \tag{A.3}$$

where

$$E_{mn}=-\frac{1}{8n^4\pi^4}-\frac{4}{n\pi^6}\sum_{m=0}^{\infty}(3C_{mn}+2B_{mn}), \tag{A.4}$$

and

$$F_{mn}=\frac{16}{\pi^6}\sum_{m=0}^{\infty}\frac{B_{mn}}{(2m+1)}\mathrm{e}^{-(2m+1)\pi(1-r)/\alpha}. \tag{A.5}$$

The constants B_{mn} and C_{mn} are defined in Equations (5.19–20–5.21). Note that, unlike the case with isothermal boundaries discussed in the paper, the expansion must be carried to order α in this case in order to capture the boundary layer effect.

Appendix B

Denoting $p_m\equiv(2m+1)\pi$, and $q_n\equiv 2n\pi$, and by z_j the complex roots of the equation $\sin z-z=0$ in the first quadrant, we obtain the stream function in Equation (5.34) as follows:

$$\chi=-\frac{32}{\pi^5}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{j=1}^{6}\chi_{j,mn}, \tag{B.1}$$

where

$$\chi_{1,mn} = (-1)^n \frac{A_{mn}}{8q_n} \xi(1+q_n\xi)\mathrm{e}^{-q_n\xi} \sin(q_n y), \tag{B.2}$$

$$\chi_{2,mn} = (-1)^n \frac{p_m^2 B_{mn}}{\left(p_m^2 - q_n^2\right)^2} \sin(q_n y) \left[\mathrm{e}^{-p_m\xi} - \mathrm{e}^{-q_n\xi} + \frac{p_m^2 - q_n^2}{2q_n} \xi \mathrm{e}^{-q_n\xi} \right], \tag{B.3}$$

$$\chi_{3,mn} = 2q_n^3 A_{mn} \left[\tfrac{1}{2}\mathfrak{Re}(\Phi_1''(q_n \mathrm{i})) + 2\mathfrak{Re} \sum_{j=1}^{\infty} \mathrm{e}^{\mathrm{i}z_j\xi} \frac{z_j \cos\left(\frac{z_j}{2}\right) \sin(z_j y)}{\cos(z_j)(z_j^2 + q_n^2)^3} \right], \tag{B.4}$$

$$\chi_{4,mn} = 2q_n p_m^2 B_{mn} \left[\mathfrak{Re}\Phi_2'(q_n \mathrm{i}) + \tfrac{1}{2} \frac{\cosh\left(\frac{p_m}{2}\right) \sinh(p_m y)}{(\sinh p_m - p_m)(p_m^2 - q_n^2)^2} \mathrm{e}^{-p_m\xi} \right.$$

$$\left. + 2\mathfrak{Re} \sum_{j=1}^{\infty} \mathrm{e}^{\mathrm{i}z_j\xi} \frac{z_j \cos\left(\frac{z_j}{2}\right) \sin(z_j y)}{\cos(z_j)(z_j^2 + p_m^2)(z_j^2 + q_n^2)^2} \right], \tag{B.5}$$

$$\chi_{5,mn} = -2q_n^3 A_{mn} \left[y\mathfrak{Re}\Psi_1''(q_n \mathrm{i}) + 4\mathfrak{Re} \sum_{j=1}^{\infty} \mathrm{e}^{\mathrm{i}z_j\xi} \frac{z_j \sin\left(\frac{z_j}{2}\right) y \cos(z_j y)}{\cos(z_j)(z_j^2 + q_n^2)^3} \right], \tag{B.6}$$

$$\chi_{6,mn} = -2q_n p_m^2 B_{mn} \left[2y\mathfrak{Re}\Psi_2'(q_n \mathrm{i}) + \frac{y \sinh(\frac{p_m}{2}) \cosh(p_m y)}{(\sinh p_m - p_m)(p_m^2 - q_n^2)^2} \mathrm{e}^{-p_m\xi} \right.$$

$$\left. + 4\mathfrak{Re} \sum_{j=1}^{\infty} \mathrm{e}^{\mathrm{i}z_j\xi} \frac{z_j \sin(\frac{z_j}{2}) y \cos(z_j y)}{\cos(z_j)(z_j^2 + p_m^2)(z_j^2 + q_n^2)^2} \right]. \tag{B.7}$$

Here, we have

$$\Phi_1(z) = \frac{z \cos\left(\frac{z}{2}\right) \sin(yz) \mathrm{e}^{\mathrm{i}\xi z}}{(\sin z - z)(z + \mathrm{i}q_n)^3}, \quad \Phi_2(z) = \frac{z \cos\left(\frac{z}{2}\right) \sin(yz) \mathrm{e}^{\mathrm{i}\xi z}}{(\sin z - z)(z^2 + p_m^2)(z + \mathrm{i}q_n)^2}, \tag{B.8–9}$$

$$\Psi_1(z) = \frac{z \sin\left(\frac{z}{2}\right) \cos(yz) \mathrm{e}^{\mathrm{i}\xi z}}{(\sin z - z)(z + \mathrm{i}q_n)^3}, \quad \Psi_2 = \frac{z \sin\left(\frac{z}{2}\right) \cos(yz) \mathrm{e}^{\mathrm{i}\xi z}}{(\sin z - z)(z^2 + p_m^2)(z + \mathrm{i}q_n)^2}. \tag{B.10–11}$$

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